

Linearity of Vector Products

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Introduction

Vector products, also known as “cross products,” are highly applicable within the fields of math and science. In computer science, graphics designers use vector products to compute the quantity and angle of light reflection required to project realistic three-dimensional objects on the screen. Physicists use vector products to compute quantities such as torque, angular momentum, and energy flux. Similarly, vector products can even be used to explain the rotational dynamics of how tornadoes or drain-waters spin.

Vector Products as Linear Transformations

Although vector products are extremely useful in many disciplines, their computation can become tedious and confusing using vectors in matrix form. The cross product of two vectors in \mathfrak{R}^3 is defined as

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}.$$

The cross product between an arbitrary vector \vec{x} and a set vector \vec{v} can be written as the linear transformation $T(\vec{x}) = \vec{v} \times \vec{x}$ and there exists a matrix A such that $T(\vec{x}) = A\vec{x}$.

We let \vec{v} be the fixed vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in \mathfrak{R}^3 and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any other vector in \mathfrak{R}^3 .

Using the formula given for the cross product, we compute

$$T(\vec{x}) = \vec{v} \times \vec{x} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{bmatrix}.$$

Next, we let $V = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and compute

$$V\vec{x} = V \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} ax_1 & bx_2 & cx_3 \\ dx_1 & ex_2 & fx_3 \\ gx_1 & hx_2 & ix_3 \end{bmatrix}.$$

We can now write $\vec{v} \times \vec{x} = V\vec{x}$ as

$$\begin{bmatrix} v_2 x_3 - v_3 x_2 \\ v_3 x_1 - v_1 x_3 \\ v_1 x_2 - v_2 x_1 \end{bmatrix} = \begin{bmatrix} ax_1 & bx_2 & cx_3 \\ dx_1 & ex_2 & fx_3 \\ gx_1 & hx_2 & ix_3 \end{bmatrix}.$$

We solve for the variable in the matrix A by comparing them to their corresponding components on the opposite side of the equation. We find that

$$\begin{aligned} a &= 0, \\ b &= -v_3, \\ c &= v_2, \\ d &= v_3, \\ e &= 0, \\ f &= -v_1, \\ g &= -v_2, \\ h &= v_1, \\ i &= 0. \end{aligned}$$

Since $T(\vec{x}) = \vec{v} \times \vec{x} = A\vec{x}$, using the values we found for the arbitrary matrix V , we can find a formula for A in terms of \vec{v} such that

$$A = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

In general this can be applied to the cross-product of any two vectors in \mathfrak{R}^3 such as

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ where}$$

$$T(\vec{x}) = \vec{y} \times \vec{x} = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \vec{x}. \quad (*)$$

Application

We now give a physical application of this technique. Torque τ , the rate of change of angular momentum is given by $\tau = \vec{r} \times \vec{F}$, where \vec{r} is a position vector (analogous to \vec{v}) and \vec{F} (analogous to \vec{x}) is the force applied to an object represented by \vec{r} . Plumbers can use the properties of lever arms and torque to their advantage. One such plumber uses an extended wrench to loosen a stubborn pipe joint. The

vector $\vec{r} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$ gives the position of the lever. The plumber pulls the wrench lever with a

force given by $\vec{F} = \begin{bmatrix} -1 \\ 6 \\ -10 \end{bmatrix}$ Newtons.

Applying the linear characteristics of the vector product, we know that

$$\mathbf{t} = T(\vec{x}) = A\vec{x}.$$

Using formula (*) we find the matrix A to be

$$\begin{bmatrix} 0 & -8 & 4 \\ 8 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix}.$$

Using a scientific calculator or computing by hand, we easily calculate

$$\begin{bmatrix} 0 & -8 & 4 \\ 8 & 0 & -1 \\ -4 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ -10 \end{bmatrix} = \begin{bmatrix} -88 \\ 2 \\ 10 \end{bmatrix}.$$

Thus we have found the torque is the vector $\mathbf{t} = \begin{bmatrix} -88 \\ 2 \\ 10 \end{bmatrix}$. This resultant vector is in a

form that is easy to plot and useful to find the direction and magnitude of the torque vector used to loosen the pipe joint.

Summary

Several branches of mathematics and science employ vector products. Linear algebraic techniques show that a vector product is a linear transformation. This fact extends the usefulness of the vector product in these fields by converting the product into a form that is easier to implement by hand and in technological instruments.

References

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